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Asymptotics of a Sequence of Witt Vectors

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We provide asymptotic and order information about the Witt vectors and integers d_n appearing in

$$\prod_{n\geq 1} \frac{1}{1+d_n(t^n/n!)} = (1-t)\,e^t.$$

1. INTRODUCTION

Let A be a commutive ring and let $W(A)$ be the ring of *Witt vectors* over the ring A. Let $\Lambda(A)$ be the free λ-ring. Then in [1] it is shown that

$$(q_n)_{n\geq 1} \mapsto \prod_{n\geq 1} (1-q_n t^n)^{-1}$$

defines an isomorphism between $W(A)$ and $\Lambda(A)$.

Denote

$$\prod_{n\geq 1} \frac{1}{1-q_n t^n} = \sum_{n\geq 0} h_n t^n.$$

The q_n correspond, via the characteristic map, to representations of the nth symmetric group. The character table of these representations would give formulae expressing the components of a Witt vector as a function of its "ghost components" [2, p. 352].

Denote

$$\prod_{n\geq 1} \frac{1}{1+d_n(t^n/n!)} = (1-t)\,e^t. \tag{*}$$

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The sequence $\{d_n\}_{n>1}$ gives the dimensions of these representations. In this paper we prove the following analytic result concerning the behaviour of $\{d_n\}$.

THEOREM 1. *For* $n = 2, 3, \ldots$

$$\begin{aligned} n \text{ odd} &\to d_n \leqslant (n-1)! \\ n \text{ prime} &\to d_n = (n-1)! \\ n \text{ even} &\to d_n \geqslant (n-1)! \end{aligned} \tag{1.1}$$

and

$$1 - \frac{1}{n} \leqslant \frac{d_n}{(n-1)!} \leqslant 1 + \frac{\alpha_n}{\sqrt{n}}, \tag{1.2}$$

where $\alpha_8 = \alpha_{16} = 2$, *and otherwise*, $\alpha_n = 1$.

We denote α_n by α for short.

2. Proof of (1.1) and (1.2)

On taking logarithms and expanding (*), it is easy to see $d_1 = 0$, and for $n > 1$

$$d_n = (n-1)! + \sum_{\substack{kh = n \\ k \neq 1, n}} \frac{(-1)^h}{h} \left(\frac{d_k}{k!}\right)^h n!, \tag{2.1}$$

so that $d_2 = 1$, $d_3 = 2$, $d_4 = 9$, $d_5 = 24$, $d_6 = 130$, $d_7 = 720$, and $d_8 = 8505$.

We note that $-d_n/n!$ are the coefficients of the Witt vector whose "ghost" is the unit vector $(1, 0, 0, 0, \ldots)$.

We will prove (1.1) and (1.2). Write

$$n = 2^{l_0} p_1^{l_1} \cdots p_k^{l_k},$$

where p_i is a prime for $1 \leqslant i \leqslant k$ and l_i is an integer for $0 \leqslant i \leqslant k$. Suppose, inductively, that (1.1) and (1.2) hold for all proper divisors of n, i.e., for each $n' = 2^{l'_0} p_1^{l'_1} \cdots p_k^{l'_k}$ with $l'_i \leqslant l_i$ $(0 \leqslant i \leqslant k)$ and $(l_0, \ldots, l_k) \neq (l'_0, \ldots, l'_k)$, (1.1) and (1.2) hold (if n is even, the left-hand side of (1.2) will be replaced by $d_n/(n-1)! \geqslant 1$).

We will show that (1.1) and (1.2) hold for

$$n = 2^{l_0} p_1^{l_1} \cdots p_k^{l_k}.$$

We have

$$d_n = (n-1)! + n! \sum{}'$$

with

$$\sum{}' := \sum_{\substack{0 \leqslant i_k \leqslant l_k \\ (i_0, ..., i_k) \neq (0, ..., 0)}} \cdots \sum_{0 \leqslant i_1 \leqslant l_1} \sum_{0 \leqslant i_0 \leqslant l_0 - 1} \left\{ \frac{1}{2^{l_0 - i_0} p_1^{l_1 - i_1} \cdots p_k^{l_k - i_k}} \right.$$
$$\left. \times \left(\frac{d_{2^{i_0} p_1^{i_1} \cdots p_k^{i_k}}}{(2^{i_0} p_1^{i_1} \cdots p_k^{i_k})!} \right)^{2^{l_0 - i_0} p_1^{l_1 - i_1} \cdots p_k^{l_k - i_k}} \right\}$$
$$- \sum_{\substack{0 \leqslant i_k \leqslant l_k \\ (i_0, ..., i_k) \neq (l_0, ..., l_k)}} \cdots \sum_{0 \leqslant i_1 \leqslant l_1} \frac{1}{p_1^{l_1 - i_1} \cdots p_k^{l_k - i_k}} \left(\frac{d_{2^{i_0} p_1^{i_1} \cdots p_k^{i_k}}}{(2^{i_0} p_1^{i_1} \cdots p_k^{i_k})!} \right)^{p_1^{l_1 - i_1} \cdots p_k^{l_k - i_k}} \quad (2.2)$$

(2.2)

We now show that

$$l_0 \geqslant 1 \Rightarrow \sum{}' \geqslant 0. \quad (2.3)$$

For each $\{i_1, ..., i_k\}$ let k_0 denote the largest number j with $l_j \neq i_j$. Let

$$A = p_1^{l_1 - i_1} \cdots p_{k_0 - 1}^{l_{k_0 - 1} - i_{k_0 - 1}} p_{k_0}^{l_{k_0} - i_{k_0} - 1},$$
$$B = 2^{l_0 - 1} p_1^{i_1} \cdots p_{k_0}^{i_{k_0}} p_{k_0 + 1}^{l_{k_0 + 1}} \cdots p_k^{l_k}.$$

Then $n = 2Ap_{k_0}B$, with A odd.

It is sufficient to prove that

$$\frac{1}{2A} \left(\frac{d_{Bp_{k_0}}}{(Bp_{k_0})!} \right)^{2A} \geqslant \frac{1}{Ap_{k_0}} \left(\frac{d_{2B}}{(2B)!} \right)^{Ap_{k_0}} \quad (2.4)$$

since this will allow us to match off terms in the second sum of (2.3).

If $B = 1$: (2.4) becomes

$$\frac{1}{2A} \left(\frac{d_{p_{k_0}}}{(p_{k_0})!} \right)^{2A} \geqslant \frac{1}{Ap_{k_0}} \left(\frac{d_2}{2} \right)^{Ap_{k_0}}. \quad (2.5)$$

Since $d_{p_{k_0}} = (p_{k_0} - 1)!$, $d_2 = 1$, then (2.5) becomes

$$\frac{1}{2} \left(\frac{1}{p_{k_0}} \right)^{2A} \geqslant \frac{1}{p_{k_0}} \left(\frac{1}{2} \right)^{Ap_{k_0}}. \quad (2.6)$$

When $p_{k_0} \geqslant 5$, then $(1/p_{k_0})^2 \geqslant (1/2)^{p_{k_0}}$, and (2.6) holds.

When $p_{k_0}=3$, we go back to (2.2). We have $p_{k_0}=3$ and $B=1 \Rightarrow n=2(3)^l$ (i.e., $l_0=1$, $l_1=l$, $k=1$) and (2.2) is

$$\begin{aligned}\sum{}' &= \sum_{0\leqslant i\leqslant l-1} \frac{1}{2(3)^i}\left(\frac{d_{3^{l-i}}}{3^{l-i}!}\right)^{2(3)^i} - \sum_{1\leqslant i\leqslant l} \frac{1}{3^i}\left(\frac{d_{2(3)^{l-i}}}{(2(3)^{l-i})!}\right)^{3^i} \\ &= \sum_{0\leqslant i\leqslant l-1} \left(\frac{1}{2(3^i)}\left(\frac{d_{3^{l-i}}}{3^{l-i}!}\right)^{2(3^i)} - \frac{1}{3^{i+1}}\left(\frac{d_{2(3^{l-i-1})}}{2(3^{l-i-1})!}\right)^{3^{i+1}}\right) \\ &:= \sum_{0\leqslant i\leqslant l-1} D_i. \end{aligned} \tag{2.7}$$

We will prove that

$$D_i \geqslant 0, \qquad 0\leqslant i\leqslant l-3 \tag{2.8}$$

and

$$D_{l-2}+D_{l-1}\geqslant 0. \tag{2.9}$$

When $l=1$, the right-hand side of (2.7) is $\frac{1}{18}-\frac{1}{24}>0$. Inductively, we suppose (2.8) is true for l and we will show that (2.8) holds for $l+1$. We have

$$\begin{aligned} D_i &\geqslant \frac{1}{2(3^i)}\left(\frac{1}{3^{l-i}}\right)^{2(3^i)}\left(1-\frac{1}{3^{l-i}}\right)^{2(3^i)} \\ &\quad - \frac{1}{3^{i+1}}\left(\frac{1}{2(3^{l-i-1})}\right)^{3^{i+1}}\left(1+\frac{1}{\sqrt{2(3^{l-i-1})}}\right)^{3^{i+1}} \\ &= \left(\frac{1}{3}\right)^{2(3^i)(l-i)+i}\left[\frac{1}{2}\left(1-\frac{1}{3^{l-i}}\right)^{2(3^i)} - \left(\frac{1}{2}\right)^{3^{i+1}}\left(\frac{1}{3}\right)^{3^i(l-i-3)+1}\right. \\ &\quad \left.\times\left(1+\frac{1}{\sqrt{2(3^{l-i-1})}}\right)^{3^{i+1}}\right]. \end{aligned} \tag{2.10}$$

When $0\leqslant i\leqslant l-3$, we have that

$$\begin{aligned} &\frac{1}{2}\left(1-\frac{1}{3^{l-i}}\right)^{2(3^i)} - \left(\frac{1}{2}\right)^{3^{i+1}}\left(\frac{1}{3}\right)^{3^i(l-i-3)+1}\left(1+\frac{1}{\sqrt{2(3^{l-i-1})}}\right)^{3^{i-1}} \\ &\geqslant \frac{1}{3}\left[\left(1-\frac{1}{27}\right)^{2(3^i)} - \left(\frac{1}{8}\right)^{3^i}\left(1+\frac{1}{\sqrt{18}}\right)^{3^{i+1}}\right]. \end{aligned} \tag{2.11}$$

Since

$$\left(1-\frac{1}{27}\right)^2 > \frac{1}{8}\left(1+\frac{1}{\sqrt{18}}\right)^3,$$

the right-hand side of (2.11) is positive. Thus (2.8) holds for $0 \leqslant i \leqslant l-3$. Moreover,

$$D_{l-2}+D_{l-1} = \frac{1}{2(3^{l-2})}\left(\frac{8}{81}\right)^{2(3^{l-2})} - \frac{1}{3^{l-1}}\left(\frac{13}{72}\right)^{3^{l-1}} + \frac{1}{2(3^{l-1})}\left(\frac{1}{3}\right)^{2(3^{l-1})} - \frac{1}{3^l}\left(\frac{1}{2}\right)^{3^l}.$$

Since

$$-\left(\frac{1}{8}\right)^3 + \left(\frac{8}{81}\right)^2 - \left(\frac{13}{72}\right)^3 > 0,$$

we have shown that

$$D_{l-2}+D_{l-1} > 0.$$

Thus we have proved that $\sum' \geqslant 0$ for $B=1$ and any p_{k_0}.

If $B=2$: (2.4) becomes

$$\frac{1}{2A}\left(\frac{d_{2p_{k_0}}}{(2p_{k_0})!}\right)^{2A} \geqslant \frac{1}{Ap_{k_0}}\left(\frac{d_4}{4!}\right)^{Ap_{k_0}} \tag{2.12}$$

and, since $d_4 = 9$, (2.12) becomes

$$\frac{1}{2}\left(\frac{d_{2p_{k_0}}}{(2p_{k_0})!}\right)^{2A} \geqslant \frac{1}{p_{k_0}}\left(\frac{3}{8}\right)^{Ap_{k_0}}. \tag{2.13}$$

For $p_{k_0} \geqslant 5$,

$$\left(\frac{1}{(2p_{k_0})}\right)^2 \geqslant \left(\frac{3}{8}\right)^{p_{k_0}} \quad \text{and} \quad \frac{1}{2} > \frac{1}{p_{k_0}},$$

and (2.13) holds because $d_{2p_{k_0}} \geqslant (2p_{k_0}-1)!$.

If $p_{k_0}=3$, then $A=3^l$ and $n=4(3^l)$; thus (2.3) becomes

$$\begin{aligned}\Sigma' &= \sum_{0\leqslant i\leqslant l-1}\frac{1}{4(3^i)}\left(\frac{d_{3^{l-i}}}{3^{l-i}!}\right)^{4(3^i)}+\sum_{0\leqslant i\leqslant l}\frac{1}{2(3^i)}\left(\frac{d_{2(3^{l-i})}}{2(3^{l-i})!}\right)^{2(3^i)}\\ &\quad-\sum_{1\leqslant i\leqslant l}\frac{1}{3^i}\left(\frac{d_{4(3^{l-i})}}{4(3^{l-i})!}\right)^{3^i}\\ &\geqslant \sum_{0\leqslant i\leqslant l-2}\left(\frac{1}{4(3^i)}\left(\frac{d_{3^{l-i}}}{3^{l-i}!}\right)^{4(3^i)}+\frac{1}{2(3^i)}\left(\frac{d_{2(3^{l-i})}}{2(3^{l-i})!}\right)^{2(3^i)}\right.\\ &\quad\left.-\frac{1}{3^{i+1}}\left(\frac{d_{4(3^{l-i-1})}}{4(3^{l-i-1})!}\right)^{3^{i+1}}\right)+\frac{1}{2(3^l)}\left(\frac{d_2}{2}\right)^{2(3^l)}-\frac{1}{3^l}\left(\frac{d_4}{24}\right)^{3^l}\\ &:=\sum_{0\leqslant i\leqslant l-2}D_i'+D_{l-1}'. \end{aligned}\tag{2.14}$$

We prove (2.14) by using induction yet again. By the induction assumption, we have that

$$\begin{aligned}D_i' &\geqslant \frac{1}{4(3^i)}\left(\frac{1}{3^{l-i}}\left(1-\frac{1}{3^{l-i}}\right)\right)^{4(3^i)}+\frac{1}{2(3^i)}\left(\frac{1}{2(3^{l-i})}\left(1-\frac{1}{2(3^{l-i})}\right)\right)^{2(3^i)}\\ &\quad-\frac{1}{3^{i+1}}\left(\frac{1}{4(3^{l-i-1})}\left(1+\frac{1}{\sqrt{4(3^{l-i-1})}}\right)\right)^{3^{i+1}}.\end{aligned}\tag{2.15}$$

When $0\leqslant i\leqslant l-2$,

$$\begin{aligned}&\left(\frac{1}{2(3^{l-i})}\left(1-\frac{1}{2(3^{l-i})}\right)\right)^2-\left(\frac{1}{4(3^{l-i-1})}\left(1+\frac{1}{\sqrt{4(3^{l-i-1})}}\right)\right)^3\\ &\quad=\frac{1}{3^{2l-2i}}\left(\frac{1}{4}\left(1-\frac{1}{2(3^{l-i})}\right)^2-\frac{1}{64}\left(\frac{1}{3}\right)^{l-i-3}\left(1+\frac{1}{\sqrt{4(3^{l-i-1})}}\right)^3\right)\\ &\quad\geqslant\frac{1}{3^{2l-2i}}\left(\frac{1}{4}\left(1-\frac{1}{18}\right)^2-\frac{3}{64}\left(1+\frac{1}{\sqrt{12}}\right)^3\right)>0.\end{aligned}$$

Moreover

$$\frac{1}{2(3^l)}\left(\frac{d_2}{2}\right)^{2(3^l)}-\frac{1}{3^l}\left(\frac{d_4}{24}\right)^{3^l}>0.$$

Thus we have proved (2.3) holds for $B=2$.

If $B\geqslant 3$: By the induction assumption, if $l_0\geqslant 2$, so that B is even, $d_{Bp_{k_0}}\geqslant(Bp_{k_0}-1)!$, and $d_{2B}\leqslant(1+\alpha/\sqrt{2B})(2B-1)!$ (where $\alpha=1$ unless

$B=4$ or 8, and $\alpha=2$ when $B=4$ or 8) because $Bp_{k_0}<n$ and $2B<n$. Then we have

$$\frac{1}{2A}\left(\frac{d_{Bp_{k_0}}}{(Bp_{k_0})!}\right)^{2A}\geqslant\frac{1}{2A}\left(\frac{1}{Bp_{k_0}}\right)^{2A}$$

and

$$\frac{1}{Ap_{k_0}}\left(\frac{d_{2B}}{(2B)!}\right)^{Ap_{k_0}}\leqslant\frac{1}{Ap_{k_0}}\left(\frac{1}{2B}\left(1+\frac{\alpha}{\sqrt{2B}}\right)\right)^{Ap_{k_0}}.$$

If $l_0=1$, then B is odd. We have that

$$\frac{d_{Bp_{k_0}}}{Bp_{k_0}!}\geqslant\frac{1}{Bp_{k_0}}\left(1-\frac{1}{Bp_{k_0}}\right).$$

Now we reduce (2.4) to

$$\frac{1}{2A}\left(\frac{1}{Bp_{k_0}}\left(1-\frac{\sigma}{Bp_{k_0}}\right)\right)^{2A}\geqslant\frac{1}{Ap_{k_0}}\left(\frac{1}{2B}\left(1+\frac{\alpha}{\sqrt{2B}}\right)\right)^{Ap_{k_0}}, \tag{2.16}$$

where $\sigma=1$ if $l_0=1$ and $\sigma=0$ if $l_0\geqslant 2$.

Let

$$f(p)=\ln\left(\frac{1}{2}\left(\frac{1}{Bp}\left(1-\frac{\sigma}{Bp}\right)\right)^{2A}\Big/\frac{1}{p}\left(\frac{1}{2B}\left(1+\frac{\alpha}{\sqrt{2B}}\right)\right)^{Ap}\right).$$

We have that

$$f'(p)=\frac{1-2A}{p}+A\ln\frac{2B}{1+\alpha/\sqrt{2B}}+2A\frac{\sigma}{p(Bp-\sigma)}>0,$$

since $A>1$ and $B\geqslant 3$. It follows that $f(p)$ is an increasing function of p. Moreover

$$f(5)=(1-2A)\ln 5-\ln 2+5A\ln\frac{2B}{1+\alpha/\sqrt{2B}}-2+A\ln\frac{B}{1-\alpha/\sqrt{5B}}>0.$$

Also

$$f(3)=(1-2A)\ln 3-\ln 2+3A\ln\frac{2B}{1+\alpha/\sqrt{2B}}-2+A\ln\frac{B}{1-\sigma/\sqrt{3B}}>0$$

provided $B \geqslant 6$ if $\alpha = 2$ and $B \geqslant 4$ if $\alpha = 1$. Thus we have that $f(p) > 0$ for $p \geqslant 3$ except $B = 4$, $\alpha = 2$, and $p = 3$. (Note: $\alpha = 2$ only for $n = 4$ or 8, i.e., $B = 2$ or 4.)

When $B = 4$, $\alpha = 2$, and $p = 3$ then $n = 8(3^l)$. In this case $\sum'$ becomes

$$\sum{}' = \sum_{0 \leqslant i \leqslant l-1} \frac{1}{8(3^i)} \left(\frac{d_{3^{l-i}}}{3^{l-i}!} \right)^{8(3^i)} + \sum_{0 \leqslant i \leqslant l} \frac{1}{4(3^i)} \left(\frac{d_{2(3^{l-i})}}{2(3^{l-i})!} \right)^{4(3^i)} + \sum_{0 \leqslant i \leqslant l} \frac{1}{2(3^i)} \left(\frac{d_{4(3^{l-i})}}{4(3^{l-i})!} \right)^{2(3^i)} - \sum_{1 \leqslant i \leqslant l} \frac{1}{3^i} \left(\frac{d_{8(3^{l-i})}}{8(3^{l-i})!} \right)^{3^i}. \tag{2.17}$$

Since f or $i \leqslant l-1$,

$$\left(\frac{d_{4(3^{l-i})}}{4(3^{l-i})!} \right)^{2(3^i)} - \left(\frac{d_{8(3^{l-i-1})}}{8(3^{l-i-1})!} \right)^{3^{i+1}} \geqslant \left(\frac{1}{4(3^{l-i})} \left(1 - \frac{1}{4(3^{l-i})} \right) \right)^{2(3^i)} - \left(\frac{1}{8(3^{l-i-1})} \left(1 + \frac{1}{\sqrt{8(3^{l-i-1})}} \right) \pi \right)^{\Theta^{i+1}}, \tag{2.18}$$

and

$$\frac{1}{16} \left(1 - \frac{1}{4(3^{l-i})} \right)^2 - \frac{1}{512} \left(\frac{1}{3} \right)^{l-i-3} \left(1 + \frac{1}{\sqrt{8(3^{l-i-1})}} \right)^3 > 0. \tag{2.19}$$

Thus the right-hand side of (2.17) is positive. We have proved (2.3) for $B \geqslant 3$. This completes the proof of (1.1).

We now prove (1.2). If n is odd, then we have

$$\frac{d_n}{(n-1)!} \leqslant 1.$$

We will prove that

$$\sum_{\substack{kh = n \\ k \neq 1, n}} \frac{1}{h} \left(\frac{d_k}{k!} \right)^h n \leqslant \frac{1}{n}, \tag{2.20}$$

when n is odd. Since $kh = n$, all of h and k we discuss are odd. Using induction again, we suppose that

$$|d_k - (k-1)!| \leqslant \frac{1}{k}, \tag{2.21}$$

for $k<n$ and k odd. Thus

$$\sum_{\substack{kh=n\\k\neq 1,n}} \frac{1}{h}\left(\frac{d_k}{k!}\right)^h n^2 \leqslant \sum_{\substack{kh=n\\k\neq 1,n}} \frac{1}{h}\left(\frac{1}{k}\right)^h n^2$$

$$= \sum_{\substack{kh=n\\k\neq 1,n}} \frac{h}{k^{h-2}}. \tag{2.22}$$

We will prove that

$$\frac{h}{k^{h-2}} \leqslant \frac{1}{k^2} \tag{2.23}$$

for $h\geqslant 5$ and $k\geqslant 5$.

Since

$$\left(\frac{\ln h}{h-4}\right)' = \frac{1-4/h-\ln h}{(h-4)^2} < 0 \qquad \text{if} \quad h\geqslant 5$$

(here $'$ denotes derivative), then

$$\frac{\ln h}{h-4} \leqslant \ln 5.$$

Thus

$$h \leqslant k^{h-4} \qquad \text{if} \quad k\geqslant 5.$$

If $h=3$,

$$\sum_{\substack{kh=n\\k\neq 1,n\\h=3}} \frac{h}{k^{h-2}} = \frac{3}{(n/3)} = \frac{9}{n}.$$

If $k=3$,

$$\sum_{\substack{kh=n\\k\neq 1,n\\k=3}} \frac{h}{k^{h-2}} = \frac{3n}{3^{n/3}}.$$

Then

$$\sum_{\substack{kh=n\\k\neq 1,n}} \frac{h}{k^{h-2}} \leqslant \sum_{i=5}^{\infty} \frac{1}{i^2} + \frac{9}{n} + \frac{3n}{3^{n/3}} \leqslant 1, \qquad \text{if} \quad n\geqslant 16.$$

If $n \leqslant 15$, since $k|n$ and $h|n$, then $3|n$ if $k=3$ or $h=3$. Thus $n=9$ or 15. For those two values of n, it is easy to check that the right-hand side of (2.22) $\leqslant 1$.

If n is even, then we will prove that (1.2) holds by induction. Supposing (1.2) true for any integer less than n, we will prove (1.2) true for n. Since $d_n/(n-1)! \geqslant 1$, it is sufficient to prove that

$$\left| \sum_{\substack{kh=n \\ k \neq 1, n}} \frac{(-1)^h}{h} \left(\frac{d_k}{k!}\right)^h n \right| \leqslant \frac{1}{\sqrt{n}}. \tag{2.24}$$

Write

$$f(k, h) = \frac{1}{h} \left(\frac{d_k}{k!}\right)^h (hk)^{3/2}.$$

Then (2.24) is

$$\sum_{\substack{kh=n \\ k \neq 1, n \\ h \text{ even}}} f(k, h) - \sum_{\substack{kh=n \\ k \neq 1, n \\ h \text{ odd}}} f(k, h) \leqslant 1. \tag{2.25}$$

To begin with we prove that for $n \geqslant 6$ and h even,

$$f(k, h) \leqslant 1/k^2. \tag{2.26}$$

First, we have that

$$f(2, h) = \left(\frac{1}{2}\right)^h 2^{3/2} h^{1/2} < \frac{1}{4}$$

for $h \geqslant 5$.

For $k=3$, we have that

$$f(3, h) = 3^{-h+3/2} h^{1/2} < \frac{1}{9}.$$

Moreover,

$$f(4, h) = \left(\frac{3}{8}\right)^h 8h^{1/2} < \frac{1}{16}.$$

For $k \geqslant 5$ and $h \geqslant 6$, we will prove

$$f(k, h) \leqslant 1/k^2 \tag{2.27}$$

as follows.

By the previous induction assumption in (1.2), it is sufficient to show that

$$\frac{1}{h}\left(\frac{(k-1)!+(k-1)!\,\alpha/\sqrt{k}}{k!}\right)^h (hk)^{3/2} \leqslant \frac{1}{k^2}. \tag{2.28}$$

We have that the left-hand side of (2.28) equals

$$\frac{h^{1/2}(\sqrt{k}+\alpha)^h}{k^{3(h-1)/2}}.$$

First we will show that, for $k \geqslant 5$ and $h \geqslant 8$,

$$\frac{h^{1/2}(\sqrt{k}+\alpha)^h}{k^{3(h-1)/2}} \leqslant \frac{1}{k^2}. \tag{2.29}$$

Now (2.29) is true if and only if

$$\frac{(\ln h)/2 + h\ln(\sqrt{k}+\alpha)}{(3/2)\,h-(7/2)} - \ln k \leqslant 0. \tag{2.30}$$

The left-hand side of (2.30) is a decreasing function of h for $h \geqslant 6$. Thus it is less than

$$\frac{\ln 6 + 12\ln\sqrt{k}+\alpha}{11} - \ln k. \tag{2.31}$$

Now (2.31) is a decreasing function of k and it is negative for $k=5$ when $\alpha=1$ and $k=6$ when $\alpha=2$. So (2.29) holds for $k \geqslant 5$, $h \geqslant 6$, after checking directly that it holds for $f(8, 6)$, $f(16, 6)$. Thus (2.9) holds for $k \geqslant 5$.

Now we have that

$$\sum_{\substack{kh=n \\ k\neq 1,n \\ h \text{ even}}} f(k,h) \leqslant \sum_{i=2}^{\infty} \frac{1}{i^2} + f\left(\frac{n}{2}, 2\right) + f\left(\frac{n}{4}, 4\right). \tag{2.32}$$

Finally, we discuss the last two terms on the right-hand side of (2.32). By the inductive hypothesis,

$$f\left(\frac{n}{2}, 2\right) \leqslant \frac{2^{1/2}(\sqrt{n/2}+\alpha)^2}{(n/2)^{3/2}}.$$

Since $f(n/2, 2)$ is a decreasing function of n and $f(35, 2) \leqslant 0.32082$ ($\alpha = 1$ if $n \neq 8$ or 16) it follows that

$$f(n/2, 2) \leqslant 0.32082 \qquad \text{for} \quad n \geqslant 72. \tag{2.33}$$

$$f\left(\frac{n}{4}, 4\right) = \frac{2(\sqrt{n/4}+1)^4}{(n/4)^{4.5}} < 0.003393 \qquad \text{for} \quad n \geqslant 72. \tag{2.34}$$

Thus the right-hand side of $(2.13) < 0.645 + 0.32082 + 0.003393 < 1$ for $n \geqslant 64$. We have shown that (1.2) holds provided we check it does for $n \leqslant 63$.

This is true except with $\alpha = 1$ except for $n = 8$ and 16 and $[d_8/(7!) = 27/16$ and $d_{16}/(15!) = 2955/2048]$ and so (1.1), (1.2) hold.

In particular

$$\lim_{n \to \infty} \frac{d_n}{(n-1)!} = 1.$$

3. Two Examples

We have

$$d_{100}/99! = \frac{1{,}437{,}875{,}310{,}019{,}956{,}682{,}521{,}937{,}269{,}338{,}988{,}093}{1{,}407{,}374{,}883{,}553{,}280{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$
$$= 1.02167\ldots.$$

As the asymptotic ensures this quantity is less than 1.1 (but it is larger than 1.01).

Similarly,

$$d_{99}/98! = \frac{23{,}433{,}924{,}589{,}424{,}318{,}014{,}557{,}665{,}728}{23{,}454{,}932{,}070{,}382{,}151{,}157{,}711{,}285{,}129}$$
$$= 0.999104\ldots.$$

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